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Counterexamples in parameter identification problem of the fractal interpolation functions ☆

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Abstract

Fractal interpolation functions provide a new method to model experimental data. Dalla and Drakopoulos got some conditions that a vertical scaling factor must obey to model effectively an arbitrary function (J. Approx. Theory 101 (1999) 289). In this paper, we present certain counterexamples to show that the converse does not hold.

Keywords: Parameter identification; Fractal interpolation function

1. Introduction

Let $\{x_0, x_1, \dots, x_N\}$ be a partition of $I = [x_0, x_N]$ (i.e., $x_0 < x_1 < \cdots < x_N$). Let $f_0, f_1, \dots, f_N$ be some given real numbers. For $n = 1, 2, \dots, N$, we define an affine map ω_n as

$$\omega_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_n & 0 \\ c_n & s_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_n \\ e_n \end{bmatrix}, \tag{1}$$

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where the real numbers a_n, c_n, s_n, d_n and e_n are chosen such that $|s_n|<1$ and such that

$$\omega_n \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ f_{n-1} \end{bmatrix} \quad \text{and} \quad \omega_n \begin{bmatrix} x_N \\ f_N \end{bmatrix} = \begin{bmatrix} x_n \\ f_n \end{bmatrix}. \tag{2}$$

We call such s_n vertical scaling factors.

Let $K = I \times \mathbf{R}$. We can define a set map $W : H(K) \to H(K)$ as

$$W(E) = \bigcup_{n=1}^{N} \omega_n(E) \quad \text{for all } E \in H(K), \tag{3}$$

where $H(K)$ is the metric space of all nonempty compact subsets of K with respect to the Hausdorff distance. It follows as in [1] that there exists a unique attractor $G \in H(K)$. Furthermore, G is the graph of a continuous function $f : [x_0, x_N] \to \mathbf{R}$ which obeys $f(x_i) = f_i,\ i = 0, 1, \ldots, N$. We call such a function an affine fractal interpolation function or AFIF for short. In order to determinate the vertical scaling factors more effectively, Dalla and Drakopoulos got the following theorem and corollaries.

Theorem 1.1 (Dalla and Drakopoulos [2, Theorem 3]). *The graph of an* AFIF *remains within a given rectangle $R = I \times [a, b]$ if and only if the vertical scaling factors s_n obey*

$$s_n^{\min} \leqslant s_n \leqslant s_n^{\max} \tag{4}$$

and $|s_n| < 1$, where

$$s_n^{\max} = \min\left\{ \frac{b - f_n}{b - f_N}, \frac{b - f_{n-1}}{b - f_0}, \frac{a - f_n}{a - f_N}, \frac{a - f_{n-1}}{a - f_0} \right\},$$

$$s_n^{\min} = \max\left\{ \frac{a - f_{n-1}}{b - f_0}, \frac{a - f_n}{b - f_N}, \frac{b - f_{n-1}}{a - f_0}, \frac{b - f_n}{a - f_N} \right\},$$

for $n = 1, 2, \ldots, N$. These bounds are the best possible.

Corollary 1.1 (Dalla and Drakopoulos [2, Corollary 1]). *The graph of an* AFIF *remains within a given rectangle $R = I \times [a, b]$ with $f_0 = f_N$ if and only if the vertical scaling factors s_n obey*

$$s_n^{\min} \leqslant s_n \leqslant s_n^{\max}$$

and $|s_n| < 1$, where

$$s_n^{\max} = \min\left\{ \frac{b - \max\{f_{n-1}, f_n\}}{b - f_0}, \frac{a - \max\{f_{n-1}, f_n\}}{a - f_0} \right\},$$

$$s_n^{\min} = \max\left\{ \frac{a - \min\{f_{n-1}, f_n\}}{b - f_0}, \frac{b - \min\{f_{n-1}, f_n\}}{a - f_0} \right\},$$

for $n = 1, 2, \ldots, N$. These bounds are the best possible.

Corollary 1.2 (Dalla and Drakopoulos [2, Corollary 2]). *The graph of an* AFIF *remains within a given strip* $S = I \times [a, \infty)$ *with* $f_0 = f_N$ *if and only if the vertical scaling factors* s_n *obey*

$$0 \leqslant s_n \leqslant s_n^{\max}$$

and $|s_n| < 1$, *where*

$$s_n^{\max} = \frac{a - \max\{f_{n-1}, f_n\}}{a - f_0},$$

for $n = 1, 2, \ldots, N$. *These bounds are the best possible.*

However, we will present some counterexamples to show that the theorem and corollaries are only valid for the one direction.

Remark 1.1. From the proof of Theorem 1.1, we can see that Dalla and Drakopoulos obtained the following useful result:

Let $M_n(x, y) = c_n x + s_n y + e_n,\ n = 1, 2, \ldots, N$, then $s_n^{\min} \leqslant s_n \leqslant s_n^{\max}$ if and only if

$$a \leqslant M_n(x, y) \leqslant b, \quad \text{for all } (x, y) \in R = I \times [a, b].$$

2. Counterexamples

Let h_f be a function which is linear on each $[x_{i-1}, x_i],\ i = 1, 2, \ldots, N$ and $h_f(x_i) = f_i,\ i = 0, 1, \ldots, N$. Let b_f be a function which is linear on $[x_0, x_N]$ and $b_f(x_0) = f_0,\ b_f(x_N) = f_N$. For $n = 1, 2, \ldots, N$, we define an affine map ω'_n as

$$\omega'_n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_n(x) \\ F_n(x, y) \end{bmatrix} = \begin{bmatrix} a_n x + d_n \\ s_n y + h_f(L_n(x)) - s_n b_f(x) \end{bmatrix}. \tag{5}$$

From [1], we can get the following lemma.

Lemma 2.1. *For any* $(x, y) \in [x_0, x_N] \times \mathbf{R}$, $\omega_n \begin{bmatrix} x \\ y \end{bmatrix} = \omega'_n \begin{bmatrix} x \\ y \end{bmatrix},\ n = 1, 2, \ldots, N.$

Let Ω be the code space $\{\omega = (i_0, i_1, \ldots, i_k, \ldots) \mid i_k \in \{0, 1, \ldots, N-1\}\}$. Define a shift operator σ on $\Omega : \Omega \to \Omega$ by $\sigma\omega = (i_1, i_2, \ldots)$, where $\omega = (i_0, i_1, i_2, \ldots)$. Define an operator $\psi : \Omega \to [x_0, x_N]$ by $\psi(\omega) = x_{i_0} + \sum_{k=1}^{\infty}\{(x_{i_k} - x_0) \bullet \prod_{l=0}^{k-1} \frac{x_{i_l+1} - x_{i_l}}{x_N - x_0}\}$, where $\omega = (i_0, i_1, i_2, \ldots)$. Then ψ is continuous and onto. Define $s_{\omega(j)} = s_{i_0+1} s_{i_1+1} \cdots s_{i_{j-1}+1}$.

Proposition 2.1. *Let f be the* AFIF *determined by* (1)–(2), then

$$f(\psi(\omega)) = \sum_{j=1}^{\infty} s_{\omega(j)} (h_f - b_f)(\psi(\sigma^j \omega)) + h_f(\psi(\omega)), \quad \omega \in \Omega.$$

Proof. By Lemma 2.1, f is the AFIF determined by (5), thus

$$(L_n(x), F_n(x, f(x))) = (L_n(x), f(L_n(x))).$$

Hence,

$$F_n(x, f(x)) = f(L_n(x)).$$

By (5), we have

$$\begin{aligned} f(x) &= F_n(L_n^{-1}(x), f(L_n^{-1}(x))) \\ &= s_n f(L_n^{-1}(x)) - s_n b_f(L_n^{-1}(x)) + h_f(x). \end{aligned} \tag{6}$$

Let $\omega \in \Omega$, we can rewrite (6) as

$$f(\psi(\omega)) = s_{\omega(1)} f(\psi(\sigma\omega)) - s_{\omega(1)} b_f(\psi(\sigma\omega)) + h_f(\psi(\omega)).$$

By induction, we have

$$\begin{aligned} f(\psi(\omega)) = &\, s_{\omega(m)}(f - b_f)(\psi(\sigma^m\omega)) \\ &+ \sum_{j=1}^{m-1} s_{\omega(j)}(h_f - b_f)(\psi(\sigma^j\omega)) + h_f(\psi(\omega)). \end{aligned} \tag{7}$$

Let $m \to \infty$, we can get the proposition since $|s_i| < 1$ for all i. □

This proposition is a generalization of Theorem 1 in [3] which is the base of an algorithm given by Ruan and Sha to solve the inverse problem of FIF.

Example 2.1. Let $I = [0, 1]$, $a = -1, b = 1$ and let $\{(0, 0), (0.5, 0.8), (1, 0)\}$ be a given set of data. Let $s_1 = 0.5, s_2 = 0$. Then we have the following

Proposition 2.2. *Let f be the* AFIF *in Example* 2.1. *Then for any $n \in \mathbf{N}$, $f(x)$ is linear on* $[1/2^n, 1/2^{n-1}]$.

Proof. Let $\Omega_n = \{\omega = (i_0, i_1, i_2, \ldots) \mid i_j = 0 \text{ for } j < n-1,\ i_{n-1} = 1\}$ be a subset of Ω. Then for each $x \in [1/2^n, 1/2^{n-1}]$, there exists $\omega \in \Omega_n$ such that $x = \psi(\omega)$.

Let $\omega = (i_0, i_1, i_2, \ldots) \in \Omega_n$, then $i_{n-1} = 1$. Hence $s_{i_{n-1}+1} = 0$ and $s_{\omega(n)} = 0$. From (7), we have

$$f(\psi(\omega)) = \sum_{j=1}^{n-1} s_{\omega(j)}(h_f - b_f)(\psi(\sigma^j\omega)) + h_f(\psi(\omega)).$$

Hence, when x is restricted on $[1/2^n, 1/2^{n-1}]$, it follows that

$$f(x) = \sum_{j=1}^{n-1} s_1^j (h_f - b_f)(L_{i_{j-1}+1}^{-1} \circ L_{i_{j-2}+1}^{-1} \circ \cdots \circ L_{i_0+1}^{-1}(x)) + h_f(x)$$

$$= \sum_{j=1}^{n-1} s_1^j (h_f - b_f)(L_1^{-j}(x)) + h_f(x) \tag{8}$$

is a linear function. □

Thus, we can get

Proposition 2.3. *Let f be the* AFIF *in Example* 2.1, *then* $\max_{x\in[0,1]} f(x) = 0.8$.

Proof. For any $n \in \mathbf{N}$, let $\omega_n = (i_0, i_1, i_2, \ldots) \in \Omega$ satisfying $i_{n-1} = 1$ and $i_j = 0$ if $j \neq n-1$. From the definition of ψ, we have $1/2^n = \psi(\omega_n)$. Since $f_0 = f_2 = 0$, $b_f(x) = 0$ for $x \in [0,1]$. By (8),

$$f(1/2^n) = \sum_{j=1}^{n-1} \frac{1}{2^j} h_f(L_1^{-j}(1/2^n)) + h_f(1/2^n)$$

$$= \sum_{j=0}^{n-1} \frac{1}{2^j} h_f(1/2^{n-j}) = \sum_{j=0}^{n-1} \frac{1}{2^j} \frac{1}{2^{n-j-1}} f_1 = \frac{n}{2^{n-1}} f_1.$$

Since

$$f(1/2^n)/f(1/2^{n+1}) = 2n/(n+1) \geqslant 1 \quad \text{for any } n \in \mathbf{N},$$

we can see that $f(1/2) = f_1 = \max_{n\in\mathbf{N}} f(1/2^n)$.

By Proposition 2.2 and f is continuous, we have

$$\max_{x\in[0,1]} f(x) = \max\left\{ \max_{n\in\mathbf{N}} f(1/2^n), f(1) \right\} = \max\{f(1/2), f(1)\} = 0.8.$$

Thus we complete the proof. □

By Proposition 2.1, we can easily see that $f(x) \geqslant 0$ for all $x \in [0,1]$. Therefore, from Proposition 2.3, the graph of the AFIF in Example 2.1 remains within $R = I \times [a,b] = [0,1] \times [-1,1]$. However, from Theorem 1.1, the graph remains within the rectangle only if

$$s_1 \leqslant \min\left\{ \frac{b-f_1}{b-f_2}, \frac{b-f_0}{b-f_0}, \frac{a-f_1}{a-f_2}, \frac{a-f_0}{a-f_0} \right\} = 0.2.$$

Thus Theorem 1.1 is valid only for the one direction.

Since $f_0 = f_2$, we can see Example 2.1 is also a counterexample of Corollary 1.1.

Define $M_n(x,y) = c_n x + s_n y + e_n$. Let $P_1 = (x_0, a), P_2 = (x_0, b), P_3 = (x_N, a), P_4 = (x_N, b)$. From the proof of Theorem 1.1, we can see that Dalla and Drakopoulos used the following result:

$$\text{(a) } G \subset R \iff \text{(b) } a \leqslant M_n(P_i) \leqslant b, \quad i = 1,2,3,4; \; n = 1,2,\ldots,N,$$

where G is the graph of the AFIF and $R = [x_0, x_N] \times [a,b]$. Since (b) is equivalent to conditions (4), Dalla and Drakopoulos obtained Theorem 1.1.

It is clear that (b) $\Rightarrow$ (a) is true. From (1) and (2), we have

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0}, \quad c_n = \frac{f_n - f_{n-1}}{x_N - x_0} - s_n \frac{f_N - f_0}{x_N - x_0},$$

$$d_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}, \quad e_n = \frac{x_N f_{n-1} - x_0 f_n}{x_N - x_0} - s_n \frac{x_N f_0 - x_0 f_N}{x_N - x_0},$$

for $n = 1,2,\ldots,N$. Thus in Example 2.1,

$$a_1 = a_2 = 0.5, \; d_1 = 0, \; d_2 = 0.5, \; c_1 = 0.8, \; c_2 = -0.8, \; e_1 = 0, \; e_2 = 0.8.$$

Hence, $M_1(x_N, b) = M_1(1,1) = 0.8 + 0.5 = 1.3 > 1 = b$. That is, condition (b) does not hold. From our above discussion, we have showed that G remains in R, i.e., condition (a) does hold. Thus (a) $\Rightarrow$ (b) is not true.

Fig. 1 displays the graphs of $W^i(R)$, $i = 1,2,3$ and the graph of the AFIF in Example 2.1.

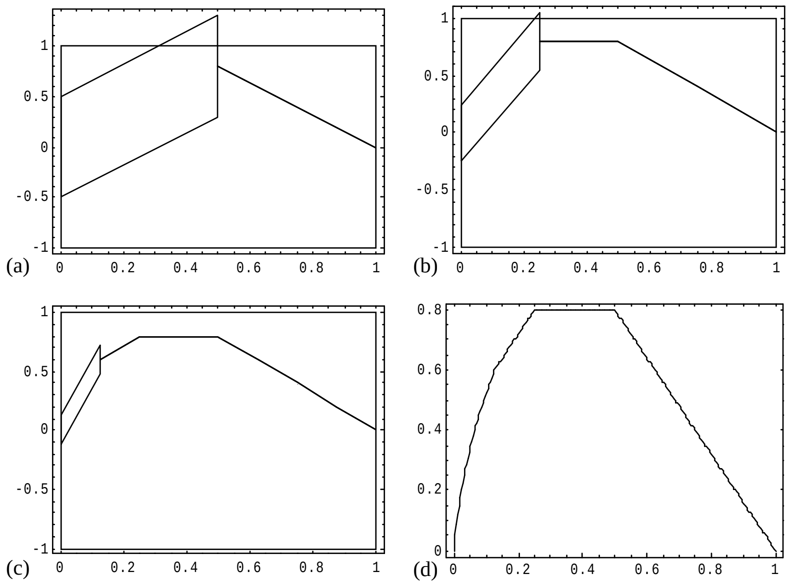

Fig. 1. (a) The graph of $W(R)$; (b) the graph of $W^2(R)$; (c) the graph of $W^3(R)$; (d) the AFIF in Example 2.1.

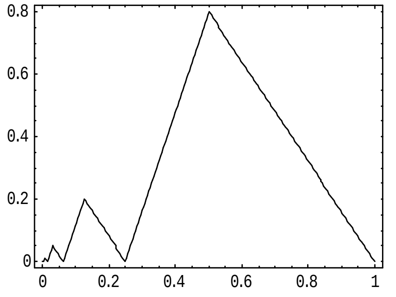

Fig. 2. The AFIF in Example 2.2.

Example 2.2. Let $I=[0,1]$ and let $\{(0,0),(0.5,0.8),(1,0)\}$ be a given set of data. Let $s_1=-0.5$, $s_2=0$. Using the same method as Example 2.1, we can see that the maximum value and the minimum value of the AFIF are 0.8 and 0, respectively. Thus, the graph of the AFIF remains within the strip $K=I\times[-1,\infty)$. However, from Corollary 1.2, the graph remains within the strip only if $s_1\geqslant 0$. This contradiction shows that Corollary 1.2 is valid only for the one direction.

Fig. 2 displays the graph of the AFIF in Example 2.2.

3. Further remarks

In Section 2, we present some examples which satisfy

$$W(R)\not\subset R \quad \text{and} \quad G\subset R,$$

where $R=I\times[a,b]$ is a given rectangle and G is the graph of an AFIF. We can easily see that $W^i(R)\subset R$ for some $i\in\mathbf{N}$ in these examples. Thus, the following question arise naturally:

Question 1. Do there exist an AFIF and a given rectangle R which satisfy the following (c) and (d)?

(c) $W^i(R)\not\subset R$, for any $i\in\mathbf{N}$,
(d) $G\subset R$.

In [4], the authors give a positive answer to this question and present some other new results. However, the following question still remains open:

Question 2. Which are sufficient and necessary conditions for s_i, $1\leqslant i\leqslant N$, to assure that the graph of an AFIF remains within a given rectangle $R=I\times[a,b]$.

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